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AN APPLICATION OF HAMILTON'S PRINCIPLE
TO DIFFRACTION OF LIGHT BY ULTRASOUND

BY

DANIEL F. WATERHOUSE
B.S.E.E., University of Texas at El Paso, 1968

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Engineering
in the Graduate Studies Program of
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ABSTRACT

AN APPLICATION OF HAMILTON'S PRINCIPLE
TO DIFFRACTION OF LIGHT BY ULTRASOUND

BY

DANIEL F. WATERHOUSE

A covariant form of Hamilton's Principle of Stationary Action is formulated and used to solve the general eiconal equation describing the wave function of light in a medium carrying ultrasound. Tensor notation is reviewed and the tensor form of Maxwell's equations is developed. Boundary equations that the field quantities must satisfy in order for the variation of Hamilton's action integral to be stationary are determined and used to form the generalized eiconal equation of geometrical optics. The rays are introduced and through a canonical transformation the eiconal for the diffracted medium is solved and plotted.

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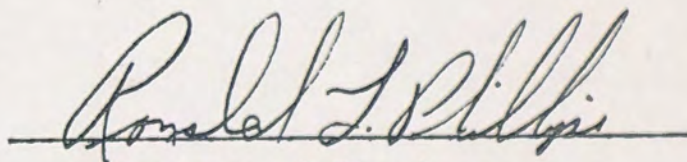

Director of Thesis

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CHAPTER I

INTRODUCTION

1.1 Method

Maxwell's equations in the normal form are not well-suited for digital computer solving. We will develop an approach which can be generalized to solve electromagnetic problems on a digital computer. The formulation is based on Hamilton's principle of stationary action. A covariant form of Hamilton's principle applied to electromagnetics provides insight into the structure of electromagnetics. It is especially valuable in problems of propagation in anisotropic and non-reciprocal media. The operational convenience is advantageous in that coordinate system transformation is greatly simplified as will be shown.

The covariance principle in physics is an example of a procedure which aims at an elimination of the subjective feature of the space-time frame of reference in the formulation of physical relations. In other words, the covariance principle or the principle of form invariance of physical laws, emphasizes a formulation of physical laws so that observers in different positions and in different states of motion, including accelerated motion, can use these same laws and have unambiguous means of correlating their observations.

The pinnacle of our development will be a covariant form of the generalized eiconal equation. In order to appreciate the significance of this development, we must understand the importance of the eiconal

equation. The eiconal is basic to the study of geometrical optics. ² It is representative of the surfaces which are the wave fronts. The orthogonal trajectories to the wavefronts are the electromagnetic rays which are curves along which the energy of the geometrical optics field flows. From this relationship we can derive the transport equations which describe the propagation of the geometrical optics field along the rays. Thus with Hamilton's Principle we will have derived geometrical optics from Maxwell's equations without the classical assumption that the wavelength approached zero.

1.2 Diffraction Problem

Sound waves, in passing through a transparent gaseous, liquid or solid medium, produce time-varying fluctuations in the density and, hence, optical index of refraction of the medium. A periodic variation of refractive index may appear to an optical beam as a diffraction grating or as a moving lens. The problem follows the typical optical diffraction scheme: the light is taken to be a parallel monochromatic beam of frequency ν traveling along the x-direction and of infinite extent in the z and y-directions. It is incident normally on a medium infinite in the z and y-directions but bounded in the x-direction as shown in Figure 1. The medium is crossed in the z-direction by progressive waves of monochromatic ultrasound with frequency ω . The optical effect of the ultrasonic wave will be represented by a sinusoidally varying refractive index. This is justified, since the density wave in the medium may be taken as sinusoidal, and for small variations the density and the refractive index are proportional. Thus, the refractive index of the medium may be expressed:

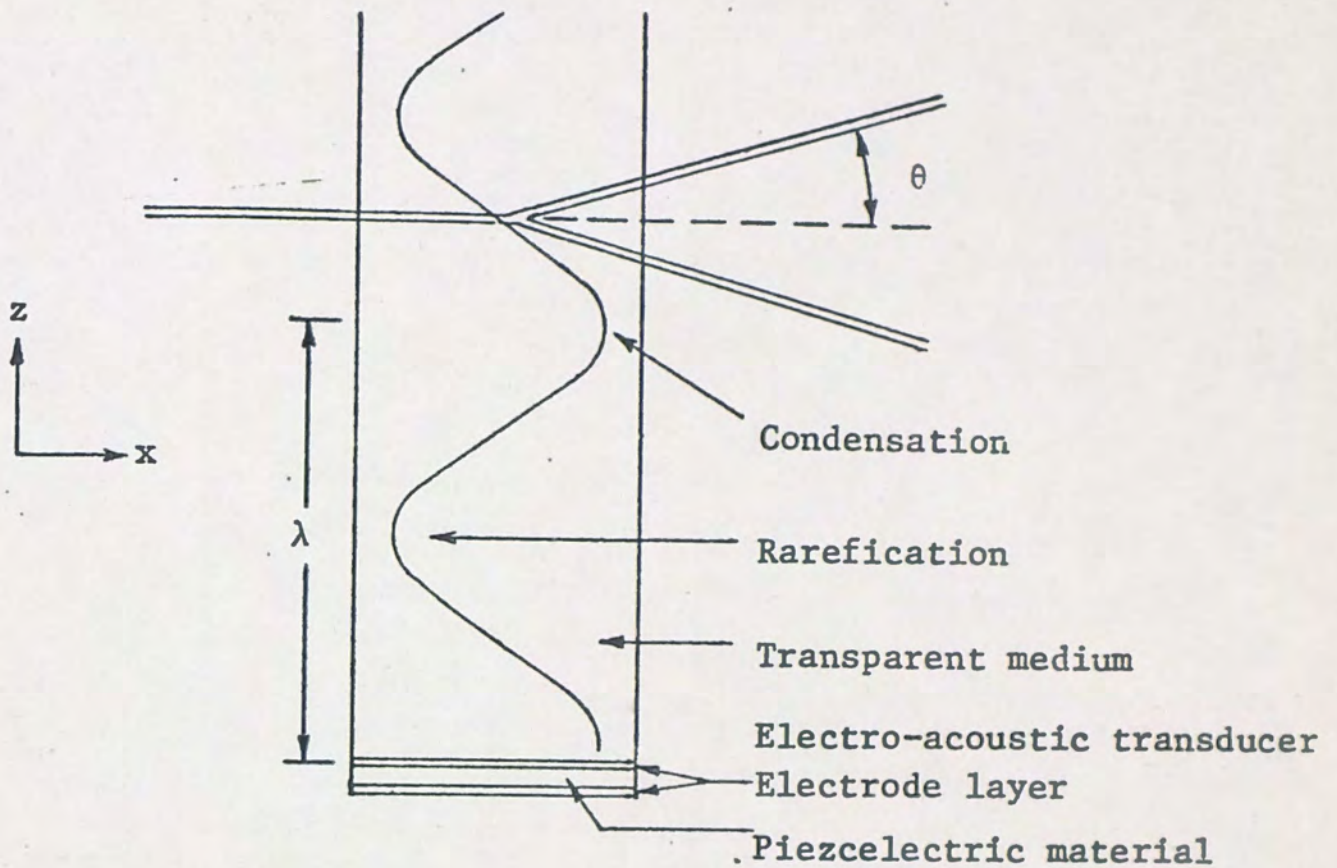


Figure 1. Ultrasonic Diffraction of Light

$$n^2 = \epsilon\mu[1 + m \cos (\omega t - \beta z)] \quad 1.2.1$$

where ϵ and μ are the permittivity and permeability constants of the medium. m is the modulation index of the ultrasound wave being defined $0 \leq m \leq 1$. Also, $\beta = 2\pi/\lambda$, where λ represents the wavelength of the ultrasound.

Since our purpose will be to illustrate a method rather than solve the problem explicitly, let us simplify the problem by using an isotropic half-space shown in Figure 2.

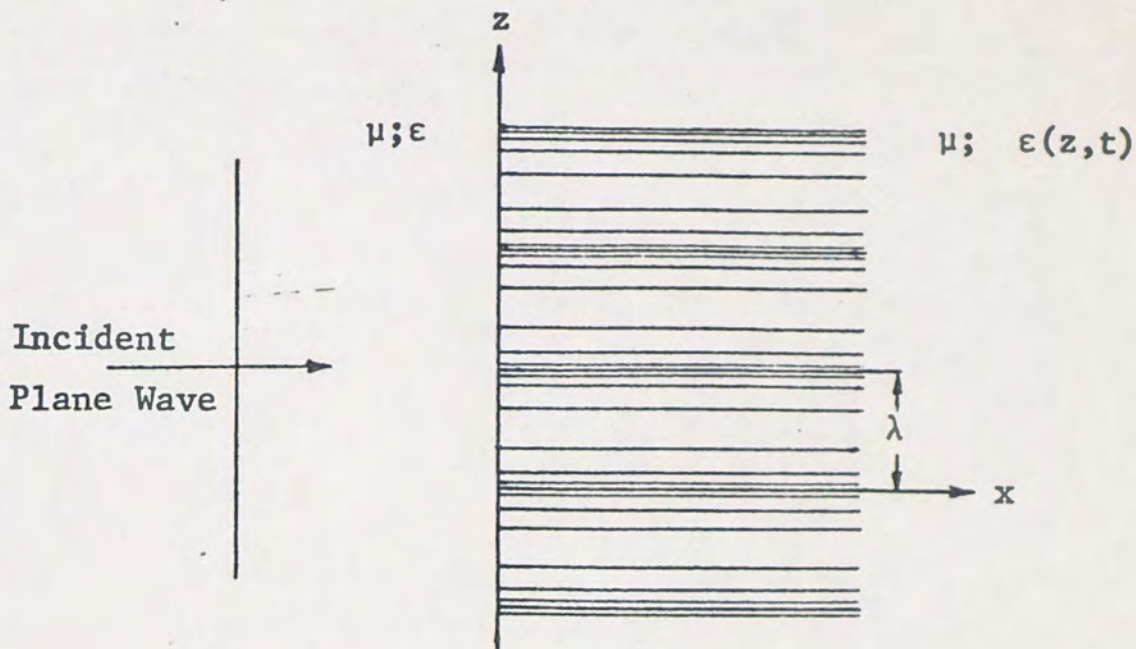


Figure 2. Sinusoidally Stratified Dielectric Half-Space

We thereby eliminate reflections and reduce the mathematical manipulations. Although the application of our method will be restricted to the geometry of a half-space, most of the techniques, interpretations, and results are applicable to less simple but more practical geometries, such as modulated slabs. The modifications required to solve these more complicated problems are mostly of a quantitative rather than a qualitative nature.

1.3 Notation

Since tensor notation will be used throughout this paper, let us briefly review the basic concepts. Let x^λ , $\lambda = 1, \dots, n$, denote a coordinate system where the Greek letter denotes a different

coordinate within that system, e.g.,

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Rectangular x^λ

$$x^1 = x$$

$$x^2 = y$$

$$x^3 = z$$

Cylindrical x^λ

$$x^1 = \rho$$

$$x^2 = \phi$$

$$x^3 = z$$

Spherical x^λ

$$x^1 = \rho$$

$$x^2 = \phi$$

$$x^3 = \theta$$

x^λ in another coordinate system will be denoted as $x^{\lambda'}$, the prime indicating a different coordinate system, i.e., a coordinate transformation $x^\lambda \rightarrow x^{\lambda'}$, where $x^{\lambda'} = x^{\lambda'}(x^\lambda)$; $\lambda, \lambda' = 1, \dots, n$. For example,

Rectangular \rightarrow Cylindrical

$$x^{\lambda'} = x^{\lambda'}(x^\lambda)$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$z = z$$

Cylindrical \rightarrow Rectangular

$$x^\lambda = x^\lambda(x^{\lambda'})$$

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

We will assume a well-behaved transformation

$$x^\lambda = x^\lambda(x^{\lambda'}) \quad 1.3.1$$

Let

$$A_{\lambda}^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \quad 1.3.2$$

and

$$A_{\lambda'}^{\lambda} = \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \quad 1.3.3$$

then

$$\begin{aligned} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \frac{\partial x^{\eta'}}{\partial x^{\eta}} &= A_{\lambda}^{\lambda'} A_{\eta}^{\eta'} \\ &= A_{\lambda}^{\lambda'} \eta'_{\eta} \end{aligned} \quad 1.3.4$$

i.e., when a product is involved, the kernel is not repeated. The

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Einstein summation convention

$$a_i b^i = \sum_{i=1}^n a_i b^i \quad 1.3.5$$

applies to a repeated index,

$$A_{\lambda}^{\lambda'} = \sum_{\lambda} A_{\lambda}^{\lambda'} A_{\lambda}^{\lambda'} \quad 1.3.6$$

We can write for consecutive transformations

$$A_{\lambda}^{\lambda'} A_{\lambda'}^{\lambda''} = A_{\lambda}^{\lambda''} \quad 1.3.7$$

which, of course, implies

$$\frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} = \frac{\partial x^{\lambda'}}{\partial x^{\lambda'}} = 1 \quad 1.3.8$$

Reciprocal transformations are expressed as

$$A_{\lambda}^{\lambda'} A_{\sigma}^{\lambda'} = A_{\sigma}^{\lambda} = \delta_{\sigma}^{\lambda} \quad 1.3.9$$

The inverse relation of $x^{\lambda} = x^{\lambda}(x^{\lambda'})$ is $x^{\lambda'} = x^{\lambda'}(x^{\lambda})$ and is related to the former by the Jacobian transformation matrix. For example, a rectangular to cylindrical transformation where $\lambda, \lambda' = 1, 2, 3$ can be expressed

$$A_{\lambda}^{\lambda'} = \begin{bmatrix} A_1^{1'} & A_2^{1'} & A_3^{1'} \\ A_1^{2'} & A_2^{2'} & A_3^{2'} \\ A_1^{3'} & A_2^{3'} & A_3^{3'} \end{bmatrix} \quad 1.3.10$$

which may be written

$$A_{\lambda}^{\lambda'} = \begin{bmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix} \quad 1.3.11$$

On evaluating the partials, we find the Jacobian to be

$$A_{\lambda}^{\lambda'} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\frac{\sin \phi}{\rho} & \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 1.3.12$$

We shall introduce time by letting $\lambda = 0$ for the time coordinate. The rectangular to cylindrical transformation can be expressed

$$\begin{bmatrix} t \\ \rho \\ \phi \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & \sin \phi & 0 \\ 0 & -\frac{\sin \phi}{\rho} & \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \quad 1.3.13$$

The short-hand notation for the determinant of the Jacobian matrix is denoted by Δ , i.e.,

$$\Delta = |A_{\lambda}^{\lambda'}| \quad 1.3.14$$

The reciprocal transformation elements are related by the expression

$$A_{\lambda}^{\lambda'} = \frac{\partial (\text{Ln} \Delta)}{\partial A_{\lambda}^{\lambda'}} \quad 1.3.15$$

$$\partial_v = \frac{\partial}{\partial x^v} \quad 1.3.16$$

Tensors are defined by the method by which they transform. They transform homogeneously. Transformation behavior of tensors is best explained through examples; if

$$T^{\sigma'} = A^{\sigma'}_{\sigma} T^{\sigma} \quad 1.3.17$$

then T is called a contravariant tensor of valence +1. Also, when

$$T_{\sigma'} = A^{\sigma}_{\sigma'} T_{\sigma} \quad 1.3.18$$

then T is called a covariant tensor of valence +1. If

$$T^{\beta'}_{\sigma'v'} = A^{\beta'\sigma v}_{\beta\sigma'v'} T^{\beta}_{\sigma v} \quad 1.3.19$$

then $T^{\beta}_{\sigma v}$ is called a mixed tensor of valence +3 contravariant in of valence +1 and covariant in σ and v of valence +2. A weighted transformation is expressed

$$T^{\lambda'}_{\sigma'} = \Delta^{-K} A^{\lambda'\sigma}_{\lambda\sigma'} T^{\lambda}_{\sigma} \quad 1.3.20$$

where $K \neq 0$. The T^{λ}_{σ} transforms as a mixed tensor of weight K . Tensors with weighted transformations are called densities.

CHAPTER II TENSOR FORM OF THE FIELDS

2.1 General Format

The electric and magnetic field vectors may be combined into four-dimensional skew-symmetric tensors according to

$$\overline{\mathbf{E}}; \overline{\mathbf{B}} \} \rightarrow F_{\nu\lambda} = -F_{\lambda\nu} \quad 2.1.1$$

$$\overline{\mathbf{D}}; \overline{\mathbf{H}} \} \rightarrow G^{\lambda\nu} = -G^{\nu\lambda} \quad 2.1.2$$

where $\nu, \lambda = 0, 1, 2, 3$.

We have

$$F_{\lambda\nu} = \begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix} \quad 2.1.3$$

and

$$G^{\lambda\nu} = \begin{bmatrix} G^{00} & G^{01} & G^{02} & G^{03} \\ G^{10} & G^{11} & G^{12} & G^{13} \\ G^{20} & G^{21} & G^{22} & G^{23} \\ G^{30} & G^{31} & G^{32} & G^{33} \end{bmatrix} \quad 2.1.4$$

The quantity $F_{\lambda\nu}$ is then constructed to transform as a covariant tensor of valence +2, i.e.,

$$F_{\lambda'\nu'} = A_{\lambda'}^{\lambda\nu} F_{\lambda\nu} \quad 2.1.5$$

The quantity $G^{\lambda\nu}$ is constructed to transform as a contravariant tensor density of valence +2, and weight +1, i.e.,

$$G^{\lambda'\nu'} = \Delta^{-(+1)} A_{\lambda\nu}^{\lambda'\nu'} G^{\lambda\nu} \quad 2.1.6$$

Then for a rectangular coordinate system, we express the electromagnetic field tensors as

$$F_{\lambda\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix} \quad 2.1.7$$

and

$$G^{\lambda\nu} = \begin{bmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{bmatrix} \quad 2.1.8$$

2.2 Constitutive Equations

Maxwell's equations in differential form for free space are

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad 2.2.1$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \quad 2.2.2$$

$$\nabla \cdot \bar{D} = \rho \quad 2.2.3$$

$$\nabla \cdot \bar{B} = 0 \quad 2.2.4$$

To allow a unique determination of the field vectors from a given distribution of currents and charges, these equations must be supplemented by relations which describe the behavior of substances under the influence of the field. These relations are known as the constitutive equations

$$\bar{D} = \epsilon \bar{E}$$

2.2.5

and

$$\bar{B} = \mu \bar{H}$$

2.2.6

The tensor expression for the linear relation of $F_{\lambda\nu}$ and $G^{\lambda\nu}$ is

$$G^{\lambda\nu} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa} \quad 2.2.7$$

where $\chi^{\lambda\nu\sigma\kappa}$ is the constitutive tensor. The transformation of $\chi^{\lambda\nu\sigma\kappa}$ is

$$\chi^{\lambda'\nu'\sigma'\kappa'} = \Delta^{-1} A_{\lambda}^{\lambda'} A_{\nu}^{\nu'} A_{\sigma}^{\sigma'} A_{\kappa}^{\kappa'} \chi^{\lambda\nu\sigma\kappa} \quad 2.2.8$$

Note that it is a contravariant tensor density of weight +1. It obeys the following relationships:

$$\chi^{\lambda\nu\sigma\kappa} = -\chi^{\nu\lambda\sigma\kappa} \quad 2.2.9a$$

$$\chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\nu\kappa\sigma} \quad 2.2.9b$$

$$\chi^{\lambda\nu\sigma\kappa} = \chi^{\sigma\kappa\lambda\nu*} \quad 2.2.9c$$

Assuming an isotropic medium, we can construct the constitutive tensor for a rectangular coordinate system to be

$$\chi^{\lambda\nu\sigma\kappa} =$$

$\sigma\kappa$	01	02	03	32	31	12
$\lambda\nu$						
01	$-\epsilon$	0	0	0	0	0
02	0	$-\epsilon$	0	0	0	0
03	0	0	$-\epsilon$	0	0	0
32	0	0	0	$\frac{1}{\mu}$	0	0
31	0	0	0	0	$\frac{1}{\mu}$	0
12	0	0	0	0	0	$\frac{1}{\mu}$

2.2.10

The general make-up of the tensor is that the two diagonal three-by-three submatrices represent the permittivity and the reciprocal of the permeability of the medium. The two off-diagonal submatrices represent optical activity and the Fresnel-Fizeau effect associated with anisotropic media. We note that

$$\chi^{0101} = -\epsilon, \quad \chi^{3232} = \frac{1}{\mu}, \quad \chi^{3223} = -\frac{1}{\mu}, \text{ etc.}$$

We will now expand the constitutive tensor equation by summing on the indices (tensor multiplication),

$$\begin{aligned} G^{\lambda\nu} &= \frac{1}{2}\chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa} \\ &= \frac{1}{2}\chi^{\lambda\nu\sigma 0} F_{\sigma 0} + \dots + \frac{1}{2}\chi^{\lambda\nu\sigma 3} F_{\sigma 3} \\ &= \frac{1}{2}\chi^{\lambda\nu 00} F_{00} + \dots + \frac{1}{2}\chi^{\lambda\nu 30} F_{30} + \dots \\ &\quad + \frac{1}{2}\chi^{\lambda\nu 03} F_{03} + \dots + \frac{1}{2}\chi^{\lambda\nu 33} F_{33}. \end{aligned} \quad 2.2.11$$

If we continue to expand on the indices λ and ν and evaluate the elements from the field and constitutive tensors (2.1.7,8 and 2.2.10), we find that the constitutive tensor equation reduces to the classical equations, i.e.,

$$\bar{D} = \epsilon \bar{E} \quad 2.2.5$$

and

$$\bar{B} = \mu \bar{H} \quad 2.2.6$$

2.3 Coordinate Transformation

Once that the constitutive tensor has been established for the particular problem and medium at hand, it may be transformed to any coordinate system, known or unknown, for the expedience of problem solution. The mechanics of this technique is elaborated with the following example, transforming from our rectangular system to the known spherical system.

We begin by forming the Jacobian transformation matrix

$$A_{\lambda}^{\lambda'} = \begin{bmatrix} A_{0}^{0'} & A_{1}^{0'} & A_{2}^{0'} & A_{3}^{0'} \\ A_{0}^{1'} & A_{1}^{1'} & A_{2}^{1'} & A_{3}^{1'} \\ A_{0}^{2'} & A_{1}^{2'} & A_{2}^{2'} & A_{3}^{2'} \\ A_{0}^{3'} & A_{1}^{3'} & A_{2}^{3'} & A_{3}^{3'} \end{bmatrix} \quad 2.3.1$$

Since we know the coordinate transformation to be

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} \\ \phi &= \tan^{-1} y/x \\ \theta &= \cos^{-1} z/\sqrt{x^2 + y^2 + z^2} \\ t &= t \end{aligned} \quad 2.3.2$$

then

$$\frac{\partial \rho}{\partial x} = \sin \theta \cos \phi, \text{ etc.}$$

The Jacobian transformation matrix from rectangular to spherical coordinates is then

$$A_{\lambda}^{\lambda'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ 0 & \frac{\cos \phi \cos \theta}{\rho} & \frac{\sin \phi \cos \theta}{\rho} & \frac{-\sin \theta}{\rho} \\ 0 & \frac{-\sin \phi}{\rho \sin \theta} & \frac{\cos \phi}{\rho \sin \theta} & 0 \end{bmatrix} \quad 2.3.3$$

The spherical constitutive tensor is formed using the constitutive tensor transformation equation (2.2.8).

We find that the determinant of the Jacobian evaluates to

$$\Delta = \frac{1}{\rho^2 \sin \theta} \quad 2.3.4$$

so that

$$\Delta^{-1} = \rho^2 \sin \theta \quad 2.3.5$$

We then proceed to find each element of the spherical constitutive tensor. We note, however, that the off-diagonal elements in this case will also be zero as in the rectangular tensor. We have, summing on κ

$$\chi^{0101'} = \rho^2 \sin \theta \left[A_{\lambda\nu\sigma\kappa}^{0101'} \chi^{\lambda\nu\sigma\kappa} + \dots + A_{\lambda\nu\sigma 3}^{0101'} \chi^{\lambda\nu\sigma 3} \right] \quad 2.3.6$$

since

$$A_0^{1'} = 0$$

then

$$A_{\lambda\nu\sigma\kappa}^{0101'} = 0$$

continuing, we sum on σ

$$\begin{aligned} \chi^{0101'} &= \rho^2 \sin \theta A_{\lambda}^{0'} A_{\nu}^{1'} \left[A_{\kappa}^{0'} (A_1^{1'} \chi^{\lambda\nu\sigma\kappa} + \dots + A_3^{1'} \chi^{\lambda\nu\sigma 3}) \right. \\ &\quad \left. + \dots + A_3^{0'} (A_1^{1'} \chi^{\lambda\nu 31} + \dots + A_3^{1'} \chi^{\lambda\nu 33}) \right] \quad 2.3.7 \end{aligned}$$

since

$$A_0^{0'} = 1$$

and

$$A_1^{0'} = A_2^{0'} = A_3^{0'} = 0$$

then summing on ν , we have

$$\begin{aligned} \chi^{0101'} &= \rho^2 \sin \theta A_{\lambda}^{0'} \left[A_1^{1'} (A_1^{1'} \chi^{\lambda 001} + \dots \right. \\ &\quad \left. + A_3^{1'} \chi^{\lambda 003}) + \dots + A_3^{1'} (A_3^{1'} \chi^{\lambda 301} + \dots + A_3^{1'} \chi^{\lambda 303}) \right] \quad 2.3.8 \end{aligned}$$

If we sum on λ , we note that

$$A_{\lambda}^{0'} = 0$$

except

$$A_0^{0'} = 1$$

Also

$$A_0^{1'} = 0$$

then

$$\begin{aligned} \chi^{0'1'0'1'} = \rho^2 \sin \theta [& A_1^{1'} (A_1^{1'} \chi^{0101} + \dots + A_3^{1'} \chi^{0103} + \dots \\ & + A_3^{1'} (A_1^{1'} \chi^{0301} + \dots + A_3^{1'} \chi^{0303})] \end{aligned} \quad 2.3.9$$

We note that

$$A_1^{1'} = \sin \theta \cos \phi$$

$$A_2^{1'} = \sin \phi \sin \theta$$

$$A_3^{1'} = \cos \theta$$

$$\chi^{0101} = \chi^{0202} = \chi^{0303} = -\epsilon$$

while all others are zero.

Then

$$\begin{aligned} \chi^{0'1'0'1'} &= \rho^2 \sin \theta [(\sin \theta \cos \phi)^2 (-\epsilon) \\ &\quad + (\sin \theta \sin \phi)^2 (-\epsilon) + \cos^2 \theta (-\epsilon)] \\ &= -\epsilon \rho^2 \sin \theta [\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta] \\ &= -\epsilon \rho^2 \sin \theta \end{aligned} \quad 2.3.10$$

The procedure is repeated for each element of the tensor, in this case, each element on the diagonal of the tensor, so that we have finally the spherical constitutive tensor:

$\chi^{\lambda\nu\alpha\kappa}$

$\sigma\kappa'$	$01'$	$02'$	$03'$	$23'$	$31'$	$12'$
$\lambda\nu'$	$-\epsilon\rho^2\sin\theta$	0	0	0	0	0
$01'$	0	$-\epsilon\rho^2\sin\theta$	0	0	0	0
$02'$	0	0	$-\epsilon/\sin\theta$	0	0	0
$03'$	0	0	0	$\frac{1}{\mu\rho^2\sin\theta}$	0	0
$23'$	0	0	0	0	$\frac{1}{\mu\sin\theta}$	0
$31'$	0	0	0	0	0	$\frac{\sin\theta}{\mu}$
$12'$	0	0	0	0	0	$\frac{\sin\theta}{\mu}$

2.3.11

2.4 Maxwell's Equations

We need to define the four-vector current density

$$J^\sigma = \begin{bmatrix} J^0 \\ J^1 \\ J^2 \\ J^3 \end{bmatrix} = \begin{bmatrix} \rho \\ J_1 \\ J_2 \\ J_3 \end{bmatrix} \tag{2.4.1}$$

where ρ is identical with charge density and J_1, J_2, J_3 are identical with the rectangular coordinates of current density. J^σ is a contra-variant tensor density of valence +1 and weight +1. It transforms as follows

$$J^{\sigma'} = \Delta^{-1} A_{\sigma}^{\sigma'} J^\sigma \tag{2.4.2}$$

We can express

$$\bar{\nabla} \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \tag{2.2.1}$$

and

$$\bar{\nabla} \cdot \bar{B} = 0 \tag{2.2.4}$$

in tensor form as

$$\partial_{[\kappa} F_{\lambda\nu]} = 0 \quad 2.4.3$$

Similarly

$$\bar{\nabla} \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \quad 2.2.2$$

and

$$\bar{\nabla} \cdot \bar{D} = \rho \quad 2.2.3$$

can be expressed

$$\partial_{\nu} G^{\lambda\nu} = J^{\lambda} \quad 2.4.4$$

where $\lambda, \nu, \kappa = 0, 1, 2, 3$. The bracket on equation (2.4.3) denotes a permutation expansion, i.e.,

$$\begin{aligned} \partial_{[\kappa} F_{\lambda\nu]} = \frac{1}{3!} [& \partial_{\kappa} F_{\lambda\nu} + \partial_{\nu} F_{\kappa\lambda} + \partial_{\lambda} F_{\nu\kappa} \\ & - \partial_{\kappa} F_{\nu\lambda} - \partial_{\nu} F_{\lambda\kappa} - \partial_{\lambda} F_{\kappa\nu}] = 0 \end{aligned} \quad 2.4.5$$

since

$$F_{\lambda\nu} = -F_{\nu\lambda}, \text{ etc.}$$

then

$$\partial_{[\kappa} F_{\lambda\nu]} = \frac{1}{3} (\partial_{\kappa} F_{\lambda\nu} + \partial_{\nu} F_{\kappa\lambda} + \partial_{\lambda} F_{\nu\kappa}) = 0 \quad 2.4.6$$

If we evaluate this equation through $\lambda, \nu, \kappa = 0, 1, 2, 3$ using our field tensor (2.1.3, 2.1.7), we find for $\lambda = 1, \nu = 2, \kappa = 3$ that

$$\frac{1}{3} [\partial_3 F_{12} + \partial_2 F_{31} + \partial_1 F_{23}] = 0 \quad 2.4.7$$

is equivalent to

$$\frac{1}{3} [\partial_3 B_3 + \partial_2 B_2 + \partial_1 B_1] = 0 \quad 2.4.8$$

which in rectangular coordinates is

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0 \quad 2.4.9$$

or

$$\bar{\nabla} \cdot \bar{B} = 0 \quad 2.2.4$$

If $\lambda = 1$, $\nu = 2$, $\kappa = 0$, we have

$$\partial_0 F_{12} + \partial_2 F_{01} + \partial_1 F_{20} = 0 \quad 2.4.10$$

which evaluates to

$$\partial_0 B_3 + \partial_2 (-E_1) + \partial_1 E_2 = 0 \quad 2.4.11$$

We can express this equation in rectangular coordinates as

$$\partial_x E_y - \partial_y E_x = -\partial_t B_z \quad 2.4.12$$

We therefore see that $\partial_{[\kappa} F_{\lambda\nu]}$ will satisfy $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$.

Expansion of $\partial_\nu G^{\lambda\nu} = J^\lambda$ in a similar fashion using the field tensor (2.1.4, 2.1.8) will show equivalence to

$$\bar{\nabla} \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + J \quad 2.2.2$$

and

$$\bar{\nabla} \cdot \bar{D} = \rho \quad 2.2.3$$

2.5 Four-Potential Wave Equation

We recall that if we substitute

$$\bar{B} = \bar{\nabla} \times \bar{A} \quad 2.5.1$$

into Maxwell's equation

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad 2.2.1$$

we have

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$$\overline{\mathbf{E}} = -\overline{\nabla}\phi - \frac{\partial \overline{\mathbf{A}}}{\partial t} \quad 2.5.2$$

We can combine the scalar potential and the vector potential into a covariant four potential tensor ϕ_λ , $\lambda = 0, 1, 2, 3$

$$\phi_\lambda = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} -\phi \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} \quad 2.5.3$$

The four-potential is related to the field tensors by

$$\begin{aligned} F_{\nu\lambda} &= 2 \partial_{[\nu} \phi_{\lambda]} \\ &= 2 \frac{1}{2!} [(\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu)] \end{aligned} \quad 2.5.4$$

then

$$F_{\nu\lambda} = \partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu \quad 2.5.5$$

If we expand on the indices and evaluate the equation on the elements of the field tensor (2.1.3, 2.1.7) we will see that the equation satisfies

$$\overline{\mathbf{B}} = \overline{\nabla} \times \overline{\mathbf{A}} \quad 2.5.1$$

and

$$\overline{\mathbf{E}} = -\overline{\nabla}\phi - \frac{\partial \overline{\mathbf{A}}}{\partial t} \quad 2.5.2$$

We will now recall our four fundamental tensor formulations in order to

put everything in terms of the four-potential:

$$F_{\nu\lambda} = \partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu \quad 2.5.5$$

$$\partial_\nu G^{\lambda\nu} = J^\lambda \quad 2.4.4$$

$$\partial_{[\kappa} F_{\lambda\nu]} = 0 \quad 2.4.3$$

$$G^{\lambda\nu} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa} \quad 2.2.7$$

If we substitute equation (2.5.5) in equation (2.2.7), we have

$$\begin{aligned} G^{\lambda\nu} &= \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} [\partial_\sigma \phi_\kappa - \partial_\kappa \phi_\sigma] \\ &= \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} \partial_\sigma \phi_\kappa - \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} \partial_\kappa \phi_\sigma \end{aligned} \quad 2.5.6$$

since

$$\chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\nu\kappa\sigma}$$

then

$$G^{\lambda\nu} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} \partial_\sigma \phi_\kappa + \frac{1}{2} \chi^{\lambda\nu\kappa\sigma} \partial_\kappa \phi_\sigma \quad 2.5.7$$

We note that the two terms on the RHS are equivalent expressions since κ and σ are just dummy variables, so that

$$G^{\lambda\nu} = \chi^{\lambda\nu\sigma\kappa} \partial_\sigma \phi_\kappa \quad 2.5.8$$

If we substitute this expression in equation (2.4.4), we have

$$\partial_\nu (\chi^{\lambda\nu\sigma\kappa} \partial_\sigma \phi_\kappa) = J^\lambda \quad 2.5.9$$

which is the tensor form of the four-potential wave equation.

We will find it worthwhile to show the equivalence of the tensor and conventional forms of the generalized wave equation. If we expand the equation for $\lambda = 0$, we have

$$\begin{aligned}
& \partial_\nu (\chi^{0\nu\sigma 0} \partial_\sigma \phi_0 + \dots + \chi^{0\nu\sigma 3} \partial_\sigma \phi_3) = J^0 \\
& \partial_0 (\chi^{00\sigma 0} \partial_\sigma \phi_0 + \dots + \chi^{00\sigma 3} \partial_\sigma \phi_3) + \dots \\
& + \partial_3 (\chi^{03\sigma 0} \partial_\sigma \phi_0 + \dots + \chi^{03\sigma 3} \partial_\sigma \phi_3) = J^0
\end{aligned} \tag{2.5.10}$$

If we inspect the rectangular constitutive tensor (2.2.10) for those elements which are zero we arrive at

$$\begin{aligned}
& \partial_1 (\chi^{0110} \partial_1 \phi_0 + \chi^{0101} \partial_0 \phi_1) + \partial_2 (\chi^{0220} \partial_2 \phi_0 + \chi^{0202} \partial_0 \phi_2) \\
& + \partial_3 (\chi^{0330} \partial_3 \phi_0 + \chi^{0303} \partial_0 \phi_3) = J^0
\end{aligned} \tag{2.5.11}$$

We now substitute the expressions for the non-zero elements from the constitutive tensor and replace the four-potential and current density connotations with the more familiar expressions to have

$$\begin{aligned}
& \partial_1 [\epsilon \partial_1 (-\phi) - \epsilon \partial_0 A_1] + \partial_2 [\epsilon \partial_2 (-\phi) - \epsilon \partial_0 A_2] \\
& + \partial_3 [\epsilon \partial_3 (-\phi) - \epsilon \partial_0 A_3] = \rho
\end{aligned} \tag{2.5.12}$$

If we replace 0, 1, 2, 3 with t, x, y, z, we have

$$(\partial_x^2 + \partial_y^2 + \partial_z^2) \phi + \partial_t (\partial_x A_x + \partial_y A_y + \partial_z A_z) = -\rho/\epsilon \tag{2.5.13}$$

which is

$$\nabla^2 \phi + \partial_t (\bar{\nabla} \cdot \bar{A}) = -\rho/\epsilon \tag{2.5.14}$$

We use the Lorentz condition

$$\bar{\nabla} \cdot \bar{A} = -\mu \epsilon \partial_t \phi \tag{2.5.15}$$

and

$$c = 1/\sqrt{\mu \epsilon} \tag{2.5.16}$$

to arrive at the conventional form of the scalar wave potential

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\rho/\epsilon \quad 2.5.17$$

The expansion on $\lambda = 1, 2, 3$ will result in the equation for the magnetic vector potential

$$\nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J} \quad 2.5.18$$

CHAPTER III

HAMILTON'S PRINCIPLE

3.1 General Theory -

Hamilton's principle of least action can be stated in the form of the action integral

$$s = \int_{t_1}^{t_2} L dt \quad 3.1.1$$

where L may represent the difference between the kinetic and potential energy of a system. We then write

$$s = \int_{t_1}^{t_2} \frac{1}{2}mv^2 dt = \int_{t_1}^{t_2} \frac{1}{2}m(\dot{x})^2 dt \quad 3.1.2$$

Assume that we have the correct function $x = f(t)$, e.g.,

$$y(\eta, t) = x(t) + \eta z(t) \quad 3.1.3$$

with a continuous first order derivative, which vanishes at the end points,

$$z(t_1) = z(t_2) = 0 \quad 3.1.4$$

Substituting equation (3.1.3) in equation (3.1.2), we have

$$s(\eta) = \int_{t_1}^{t_2} \frac{1}{2}m\dot{y}^2 dt \quad 3.1.5a$$

$$\frac{\partial s}{\partial \eta} = \int_{t_1}^{t_2} m(2)\dot{y} \frac{\partial \dot{y}}{\partial \eta} dt \quad 3.1.5b$$

$$= \int_{t_1}^{t_2} m\dot{y} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial \eta} \right) dt \quad 3.1.5c$$

Integrating by parts, we have

$$\frac{\partial s}{\partial \eta} = m\dot{y} \frac{\partial y}{\partial \eta} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial y}{\partial \eta} m \frac{\partial}{\partial t} (\dot{y}) dt \quad 3.1.6$$

but from equation (3.1.3)

$$\frac{\partial y}{\partial \eta} = z(t) \quad 3.1.7$$

then

$$\frac{\partial s}{\partial \eta} = \dot{m}y z(t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{y} \frac{\partial y}{\partial \eta} dt \quad 3.1.8$$

and by our definition of $z(t)$, equation (3.1.4),

$$\frac{\partial s}{\partial \eta} = - \int_{t_1}^{t_2} m\ddot{y} \frac{\partial y}{\partial \eta} dt \quad 3.1.9$$

Now, we apply Hamilton's principle, which says that the variation of the action integral must be stationary, i.e.,

$$\frac{\partial s}{\partial \eta} \Big|_{\eta=0} = 0 \quad 3.1.10$$

This is identical to saying

$$m\ddot{y} = 0 \quad 3.1.11$$

since by design

$$\frac{\partial y}{\partial \eta} = z(t) \neq 0 \quad 3.1.12$$

Furthermore, when $\eta = 0$, from equation (3.1.3) we have $\dot{y} = \dot{x}$

then

$$\ddot{x} = 0 \quad 3.1.13$$

and

$$\dot{x} = \text{const} = K_1 \quad 3.1.14$$

or

$$x = K_1 t + K_2 \quad 3.1.15$$

which satisfies the action integral.

We now generalize the action integral by defining L as a functional

$$L = L(x, \dot{x}, t) . \quad 3.1.16$$

If we use the relationships defined in equations (3.1.3, 3.1.4), then

$$s(\eta) = \int_{t_1}^{t_2} L(y, \dot{y}, t) dt \quad 3.1.17a$$

$$\frac{\partial s}{\partial \eta} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial L}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \eta} \right) dt \quad 3.1.17b$$

$$= \frac{\partial L}{\partial y} \frac{\partial y}{\partial \eta} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \eta} dt \quad 3.1.17c$$

If we substitute equation (3.1.7), then the first expression on the right vanishes. We have

$$\frac{\partial s}{\partial \eta} = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \eta} dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{y}} \frac{d}{dt} \left(\frac{\partial y}{\partial \eta} \right) dt \quad 3.1.18$$

Integrating by parts, we have

$$\frac{\partial s}{\partial \eta} = \frac{\partial L}{\partial \dot{y}} \frac{\partial y}{\partial \eta} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial y}{\partial \eta} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) dt \quad 3.1.19$$

and by substitution of equation (3.1.7) reduces to

$$\frac{\partial s}{\partial \eta} = - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) \frac{\partial y}{\partial \eta} dt \quad 3.1.20$$

We note that

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) \frac{\partial y}{\partial \eta} = \frac{d}{dt} \frac{\partial}{\partial (dy/dt)} + \frac{d}{dt} \left(\frac{\partial y}{\partial \eta} \right) \frac{\partial L}{\partial \dot{y}} \quad 3.1.21$$

If we substitute this expression into equation (3.1.17b), we have

$$- \frac{\partial s}{\partial \eta} = \int_{t_1}^{t_2} \frac{\partial L}{\partial y} \frac{\partial y}{\partial \eta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) \frac{\partial y}{\partial \eta} dt \quad 3.1.22$$

Now we apply Hamilton's principle to arrive at

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$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} \right) z(t) dt = 0 \quad 3.1.23$$

since by design $z(t) \neq 0$, then

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) = 0 . \quad 3.1.24$$

This is Lagrange's equation whereby the functional L is referred to as the Lagrangian.

3.2 Action Integral

Normally, the variational principle is applied to discrete systems. Electromagnetic field theory is a continuous system. We will generalize a variational technique where the discrete systems are replaced by continuous systems basing this approach on the relationship of the Lagrangian and the constitutive equations in their explicit forms. We will consider the case where we have instantaneous and local relationships among the electromagnetic field quantities.

The Lagrangian can be expressed as a volume integral

$$L = \int_{\tau} \bar{L} d\tau \quad 3.2.1$$

where \bar{L} is a functional having units of energy density. The action can then be expressed as a four-dimensional integral

$$s = \int_{t_1}^{t_2} \int_{\tau} \bar{L} d\tau dt = \int \bar{L} df \quad 3.2.2$$

where it is a scalar expression invariant under general space-time transformations. The Lagrangian density and the four-dimensional integration element transform according to the rule

$$\bar{L}(x^{K'}) = \Delta^{-1} \bar{L}(x^K) \quad 3.2.3a$$

$$df(x^{K'}) = \Delta^1 df(x^K) \quad 3.2.3b$$

Suppose the Lagrangian density is

$$\mathcal{L} = \frac{1}{4} G^{\nu\lambda} F_{\nu\lambda} \quad 3.2.4$$

If we substitute the constitutive equation

$$G^{\nu\lambda} = \frac{1}{2} \chi^{\nu\lambda\sigma\kappa} F_{\sigma\kappa} \quad 2.2.7$$

we have

$$\bar{\mathcal{L}} = 1/8 \chi^{\nu\lambda\sigma\kappa} F_{\nu\lambda} F_{\sigma\kappa} \quad 3.2.5$$

Now we substitute the four-potential forms of the tensors $F_{\nu\lambda}$ and $F_{\sigma\kappa}$

$$\begin{aligned} F_{\nu\lambda} &= (\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu) \\ F_{\sigma\kappa} &= (\partial_\sigma \phi_\kappa - \partial_\kappa \phi_\sigma) \end{aligned} \quad 2.5.5$$

so that

$$\bar{\mathcal{L}} = 1/8 \chi^{\nu\lambda\sigma\kappa} (\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu)(\partial_\sigma \phi_\kappa - \partial_\kappa \phi_\sigma) \quad 3.2.6$$

If we expand the equation as follows

$$\bar{\mathcal{L}} = 1/8 (\chi^{\nu\lambda\sigma\kappa} \partial_\nu \phi_\lambda - \chi^{\nu\lambda\sigma\kappa} \partial_\lambda \phi_\nu)(\partial_\sigma \phi_\kappa - \partial_\kappa \phi_\sigma) \quad 3.2.7$$

and substitute

$$\chi^{\lambda\nu\sigma\kappa} = -\chi^{\nu\lambda\sigma\kappa}$$

we have

$$\bar{\mathcal{L}} = 1/8 (\chi^{\nu\lambda\sigma\kappa} \partial_\nu \phi_\lambda + \chi^{\lambda\nu\sigma\kappa} \partial_\lambda \phi_\nu)(\partial_\sigma \phi_\kappa - \partial_\kappa \phi_\sigma) \quad 3.2.8$$

since λ and ν are only dummy indices, we interchange them on one term to arrive at

$$\bar{\mathcal{L}} = \frac{1}{4} \chi^{\lambda\nu\sigma\kappa} \partial_\lambda \phi_\nu (\partial_\sigma \phi_\kappa - \partial_\kappa \phi_\sigma) \quad 3.2.9$$

We then repeat this procedure to have

$$\bar{L} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} \partial_\lambda \Phi_\nu \partial_\sigma \Phi_\kappa \quad 3.2.10$$

If the medium contains current sources, the Lagrange density can be written

$$\bar{L} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} \partial_\lambda \Phi_\nu \partial_\sigma \Phi_\kappa - \Phi_\eta J^\eta \quad 3.2.11$$

The action integral is then expressed

$$S = \int \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} \partial_\lambda \Phi_\nu \partial_\sigma \Phi_\kappa df - \int \Phi_\eta J^\eta df . \quad 3.2.12$$

3.3 Variation of the Action Integral

We now will develop the equations that the electromagnetic fields must satisfy for the variation of the action integral to be stationary. The Lagrangian will comprise a system consisting of a homogeneous medium of infinite extent. Let us consider the system for a finite amount of time, from 0 to time t . The variation of the action integral encompasses both a variation of the integrand and a variation of the boundary of the region of integration. Hence, we consider the variation of the action integral based on the function of both the space-time coordinates x^λ , and the four-potential components Φ_λ .

The action integral, neglecting current sources, is

$$S = \frac{1}{2} \int \chi^{\lambda\nu\sigma\kappa} \partial_\lambda \Phi_\nu \partial_\sigma \Phi_\kappa df \quad 3.3.1$$

If we consider the action as a function of a parameter ϵ , then in order to take the variation $y = x + \epsilon$, we move the action to a new coordinate system

$$x^{\lambda'} = x^{\lambda'}(x^\lambda, \epsilon) \quad 3.3.2$$

$$\Phi = \Phi(x^\lambda(x^\lambda, \epsilon), \epsilon)$$

$$df' = \Delta df$$

then

$$s = \frac{1}{2} \int \chi^{\lambda\nu\sigma\kappa'} \partial_{\lambda'} \Phi_{\nu'} \partial_{\sigma'} \Phi_{\kappa'} \Delta^{-1} df' \quad 3.3.3$$

We need to apply Hamilton's principle, i.e.,

$$\left. \frac{\partial s}{\partial \epsilon} \right|_{\epsilon \rightarrow 0} = 0 \quad 3.3.4$$

Proceeding, we expand using

$$\frac{\partial uv}{\partial x} = \frac{\partial u}{\partial x} v + \frac{\partial v}{\partial x} u \quad 3.3.5$$

We have

$$\begin{aligned} \frac{\partial s}{\partial \epsilon} &= \frac{1}{2} \int \partial_{\epsilon} [\chi^{\lambda\nu\sigma\kappa'} \partial_{\lambda'} \Phi_{\nu'} \partial_{\sigma'} \Phi_{\kappa'} \Delta^{-1}] df \\ &= \frac{1}{2} \int \partial_{\epsilon} [\chi^{\lambda\nu\sigma\kappa'} \partial_{\lambda'} \Phi_{\nu'} \partial_{\sigma'} \Phi_{\kappa'}] \Delta^{-1} df \\ &\quad + \frac{1}{2} \int \chi^{\lambda\nu\sigma\kappa'} \partial_{\lambda'} \Phi_{\nu'} \partial_{\sigma'} \Phi_{\kappa'} \partial_{\epsilon} (\Delta^{-1}) df \end{aligned} \quad 3.3.6$$

Since $\partial_{\lambda'} \Phi_{\nu'}$ and $\partial_{\sigma'} \Phi_{\kappa'}$ are implicitly and explicitly functions of ϵ we can utilize the chain rule for partial derivatives

$$\begin{aligned} \frac{\partial s}{\partial \epsilon} &= \frac{1}{2} \int \partial_{\sigma} (\chi^{\lambda\nu\sigma\kappa'}) A_{\epsilon}^{\delta} \partial_{\lambda'} \Phi_{\nu'} \partial_{\sigma'} \Phi_{\kappa'} \Delta^{-1} df \\ &\quad + \frac{1}{2} \int \chi^{\lambda\nu\sigma\kappa'} [\partial_{\sigma} (\partial_{\lambda'} \Phi_{\nu'}) A_{\epsilon}^{\delta} + \partial_{\epsilon} (\partial_{\lambda'} \Phi_{\nu'})] \partial_{\sigma'} \Phi_{\kappa'} \Delta^{-1} df \\ &\quad + \frac{1}{2} \int \chi^{\lambda\nu\sigma\kappa'} \partial_{\lambda'} \Phi_{\nu'} [\partial_{\sigma} (\partial_{\sigma'} \Phi_{\kappa'}) A_{\epsilon}^{\delta} + \partial_{\epsilon} (\partial_{\sigma'} \Phi_{\kappa'})] \Delta^{-1} df \\ &\quad + \frac{1}{2} \int \chi^{\lambda\nu\sigma\kappa'} \partial_{\lambda'} \Phi_{\nu'} \partial_{\sigma'} \Phi_{\kappa'} \partial_{\epsilon} (\Delta^{-1}) df \end{aligned} \quad 3.3.7$$

If we change the order of indices of the second term of equation (3.3.7)

$$\frac{1}{2} \int \chi^{\sigma\kappa\lambda\nu} [\partial_{\sigma} (\partial_{\sigma'} \Phi_{\kappa'}) A_{\epsilon}^{\sigma} + \partial_{\epsilon} (\partial_{\sigma'} \Phi_{\kappa'})] \partial_{\lambda'} \Phi_{\nu'} \Delta^{-1} df$$

but since

$$\chi^{\sigma\kappa\lambda\nu} = \chi^{\lambda\nu\sigma\kappa}$$

then the second term is equal to the third term of equation (3.3.7). Let us look at equation (3.3.7) again and examine the integrands of the first term and the first part of the second and third terms, i.e.,

$$\begin{aligned} & \partial_\gamma (\chi^{\lambda\nu\sigma\kappa}) A_\epsilon^\gamma \partial_\lambda \Phi_\nu \partial_\sigma \Phi_\kappa \Delta^{-1} \\ & + \chi^{\lambda\nu\sigma\kappa} \partial_\gamma (\partial_\lambda \Phi_\nu) A_\epsilon^\gamma \partial_\sigma \Phi_\kappa \Delta^{-1} \\ & + \chi^{\lambda\nu\sigma\kappa} \partial_\lambda \Phi_\nu \partial_\gamma (\partial_\sigma \Phi_\kappa) A_\epsilon^\gamma \Delta^{-1} \end{aligned}$$

We note that this is a partial derivative expansion of the product term

$$\partial_\gamma (\chi^{\lambda\nu\sigma\kappa} \partial_\lambda \Phi_\nu \partial_\sigma \Phi_\kappa) A_\epsilon^\gamma \Delta^{-1}$$

which can be expressed as

$$\begin{aligned} \partial_\gamma (\bar{L}) A_\epsilon^\gamma \Delta^{-1} &= \partial_\gamma (\bar{L}) A_{\beta'}^\alpha B_{\alpha}^{\beta'} A_\epsilon^\gamma \\ &= \partial_{\beta'} (\bar{L}) B_{\alpha}^{\beta'} A_\epsilon^\alpha \end{aligned} \quad 3.3.8$$

where $B_{\alpha}^{\beta'}$ is the cofactor of $A_{\beta'}^\alpha$, i.e.,

$$\Delta^{-1} = A_{\beta'} B_{\alpha}^{\beta'} \quad 3.3.9$$

Also, since the second and third terms of equation (3.3.7) were found equivalent, we can combine the last part of these two terms as

$$\int \chi^{\lambda\nu\sigma\kappa} \partial_\epsilon (\partial_\sigma \Phi_\kappa) \partial_\lambda \Phi_\nu \Delta^{-1} df$$

Consider now the fourth term of equation (3.3.7),

$$\frac{1}{2} \int \chi^{\lambda\nu\sigma\kappa} \partial_\lambda \Phi_\nu \partial_\sigma \Phi_\kappa \partial_\epsilon (\Delta^{-1}) df$$

Note that

$$\partial_\epsilon (\Delta^{-1}) = B_{\alpha}^{\beta'} \partial_\epsilon A_{\beta'}^\alpha \quad 3.3.10$$

or

$$\partial_{\epsilon} (\Delta^{-1}) = B^{\beta'} \partial_{\beta'} A_{\epsilon} \quad 3.3.11$$

Now we will collect all of our alterations and express the variation as

$$\begin{aligned} \frac{\partial S}{\partial \epsilon} &= \int \chi^{\lambda \nu \sigma \kappa'} \partial_{\epsilon} (\partial_{\sigma} \Phi_{\kappa'} \partial_{\lambda} \Phi_{\nu} \Delta^{-1}) df' \\ &+ \frac{1}{2} \int \partial_{\beta'} (\chi^{\lambda \nu \sigma \kappa'} \partial_{\lambda} \Phi_{\nu} \partial_{\sigma} \Phi_{\kappa'}) B_{\alpha}^{\beta'} A_{\epsilon}^{\alpha} df' \\ &+ \frac{1}{2} \int \chi^{\lambda \nu \sigma \kappa'} \partial_{\lambda} \Phi_{\nu} \partial_{\sigma} \Phi_{\kappa'} B_{\alpha}^{\beta'} \partial_{\beta'} A_{\epsilon}^{\alpha} df' \end{aligned} \quad 3.3.12$$

If we observe that the latter two integrands are elements of a partial derivative of a product, we can express the variation as

$$\begin{aligned} \frac{\partial S}{\partial \epsilon} &= \int \chi^{\lambda \nu \sigma \kappa'} \partial_{\sigma} (\partial_{\epsilon} \Phi_{\kappa'}) \partial_{\lambda} \Phi_{\nu} \Delta^{-1} df' \\ &+ \frac{1}{2} \int B_{\alpha}^{\beta'} \partial_{\beta'} (\chi^{\lambda \nu \sigma \kappa'} \partial_{\lambda} \Phi_{\nu} \partial_{\sigma} \Phi_{\kappa'} A_{\epsilon}^{\alpha}) df' \end{aligned} \quad 3.3.13$$

If we let $\epsilon \rightarrow 0$, then

$$\frac{\partial S}{\partial \epsilon} \Big|_{\epsilon \rightarrow 0} = B_{\alpha}^{\beta'} = A_{\beta'}^{\alpha} \quad 3.3.14$$

and

$$x^{\lambda'} = x^{\lambda} \quad 3.3.15$$

then

$$\begin{aligned} \frac{\partial S}{\partial \epsilon} &= \delta S = \int \chi^{\lambda \nu \sigma \kappa} \partial_{\lambda} \Phi_{\nu} \partial_{\sigma} (\delta \Phi_{\kappa}) df \\ &+ \frac{1}{2} \int \partial_{\alpha} (\chi^{\lambda \nu \sigma \kappa} \partial_{\lambda} \Phi_{\nu} \partial_{\sigma} \Phi_{\kappa} \delta x^{\alpha}) df \end{aligned} \quad 3.3.16$$

If we observe that the integrand of the first term is an element of a partial derivative of a product, in particular

$$\partial_{\sigma} [(\chi^{\lambda \nu \sigma \kappa} \partial_{\lambda} \Phi_{\nu}) \delta \Phi_{\kappa}] = \partial_{\sigma} (\chi^{\lambda \nu \sigma \kappa} \partial_{\lambda} \Phi_{\nu}) \delta \Phi_{\kappa} + \chi^{\lambda \nu \sigma \kappa} \partial_{\lambda} \Phi_{\nu} \partial_{\sigma} (\delta \Phi_{\kappa}) \quad 3.3.17$$

We can express the variation as

$$\begin{aligned} \delta s = & \int \partial_{\sigma} [(\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu}) \delta \phi_{\kappa}] df - \int \partial_{\sigma} (\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu}) \delta \phi_{\kappa} df \\ & + \frac{1}{2} \int \partial_{\alpha} (\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu} \partial_{\sigma} \phi_{\kappa} \delta x^{\alpha}) df \end{aligned} \quad 3.3.18$$

We now apply Gauss's law which says that in any closed domain G of three dimensional space bounded by a hypersurface ψ the vector field \bar{D} on ψ is related to the total charge in G by

$$\frac{1}{4\pi} \oint_{\psi} \bar{D} \cdot \bar{n} ds = \int_{\tau} \rho d\tau \quad 3.3.19$$

where \bar{n} is the outward unit vector normal to ψ and ds is a surface element of ψ . Since G is an arbitrary closed domain, Gauss's divergence theorem says that

$$\oint_{\psi} \bar{D} \cdot \bar{n} ds = \int_{\tau} \bar{\nabla} \cdot \bar{D} d\tau \quad 3.3.20$$

Applying this theorem, we have

$$\begin{aligned} \delta s = & \oint_{M_{\sigma}} (\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu}) \delta \phi_{\kappa} ds \\ & + \frac{1}{2} \oint_{M_{\alpha}} (\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu} \partial_{\sigma} \phi_{\kappa}) \delta x^{\alpha} ds \\ & - \int \partial_{\sigma} (\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu}) \delta \phi_{\kappa} df \end{aligned} \quad 3.3.21$$

where M_{σ} and M_{α} are unit vectors normal to the hypersurface.

If we now apply Hamilton's principle

$$\delta s = 0 \quad 3.3.22$$

then

$$M_{\sigma} \chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu} = 0 \quad \kappa = 0, 1, 2, 3 \quad 3.3.23a$$

$$\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu} \partial_{\sigma} \phi_{\kappa} = 0 \quad 3.3.23b$$

$$\text{and} \quad \partial_{\sigma} (\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu}) = 0 \quad \kappa = 0, 1, 2, 3 \quad 3.3.23c$$

If we consider that the medium contains current sources, then equation (3.3.23c) becomes

$$\partial_{\sigma} (\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu}) = J^{\kappa} \quad 2.5.9$$

which is the generalized four potential wave equation developed earlier.

The other two equations are the boundary conditions which the four-potential must satisfy in a medium of infinite extent.

CHAPTER IV

EICONAL EQUATION

4.1 Generalized Eiconal Equation

The boundary equations developed from the variation of the action integral are

$$M_{\sigma} \chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu} = 0 \quad \kappa = 0, 1, 2, 3 \quad 3.3.23a$$

$$\chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu} \partial_{\sigma} \phi_{\kappa} = 0 \quad 3.3.23b$$

We recall from the four-potential development that

$$G^{\lambda\nu} = \chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu} \quad 2.5.8$$

and from the Lagrangian density formulation, equations (3.2.4, 3.2.10),

$$\frac{1}{4} G^{\lambda\nu} F_{\lambda\nu} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} \partial_{\lambda} \phi_{\nu} \partial_{\sigma} \phi_{\kappa} \quad 4.1.1$$

so that the boundary equations may be expressed

$$M_{\lambda} G^{\lambda\nu} = 0 \quad 4.1.2a$$

$$G^{\lambda\nu} F_{\lambda\nu} = 0 \quad 4.1.2b$$

If we substitute the first equation into the second, we have

$$M_{[\lambda} F_{\sigma\kappa]} = 0 \quad 4.1.2c$$

Equations (4.1.2a) and (4.1.2c) are the tensor forms of the boundary equations.

Let us stop a moment to consider the problem at hand. We have a source of electromagnetic energy, turned on at time $t = 0$. At that instant the field begins to propagate into space that is void of any other energy $t > 0$. Consider now a hypersurface which separates the space : on one side we have the field and on the other we have nothing. The surface represents a boundary, that is, we can assume that the field quantities will have a jump discontinuity on this surface.

The equation of the surface can be written

$$\Psi(x^0, x^1, x^2, x^3) = 0 . \quad 4.1.3$$

The gradient of the function Ψ can be considered a covariant vector whose components are denoted

$$(\bar{\nabla} \Psi)_{\lambda} = \partial_{\lambda} \Psi \quad g^{\lambda\sigma} \quad 4.1.4$$

Let Γ denote the magnitude of the gradient, i.e.,

$$\Gamma = | \bar{\nabla} \Psi | = (\partial_{\lambda} \Psi \quad g^{\lambda\sigma} \quad \partial_{\sigma} \Psi)^{\frac{1}{2}} \quad 4.1.5$$

where $g^{\lambda\sigma}$ is the metric tensor in rectangular coordinates

$$g^{\lambda\sigma} = \begin{bmatrix} -\frac{1}{c^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad 4.1.6$$

The constant c denotes the velocity of light. It converts time from seconds to meters for four-space representation. We now form the unit tensor

$$M_{\lambda} = \frac{\partial_{\lambda} \Psi}{\Gamma} \quad g^{\lambda\sigma} \quad 4.1.7a$$

where

$$M_{\lambda} = (M_0, M_1, M_2, M_3) \quad 4.1.7b$$

If Ψ is a function of time and one spatial coordinate only, say

$$\Psi = \Psi(x, t) \quad 4.1.8a$$

then

$$M_{\lambda} = (M_0, M_1, 0, 0) \quad 4.1.8b$$

We now proceed to develop the generalized eiconal equation. If we recall the constitutive tensor relation of the fields

$$G^{\lambda\nu} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa} \quad 2.2.7$$

then the tensor form of the boundary equations may be written

$$M_{\lambda} \chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa} = 0 \quad 4.1.9$$

and

$$M_{[\lambda} F_{\sigma\kappa]} = 0 \quad 4.1.2c$$

If we expand $M_{[\lambda} F_{\sigma\kappa]} = 0$, we have (see equation 2.4.6)

$$M_{\lambda} F_{\sigma\kappa} + M_{\sigma} F_{\kappa\lambda} + M_{\kappa} F_{\lambda\sigma} = 0 \quad 4.1.10$$

If we substitute into equation (4.1.9), we have

$$- \chi^{\lambda\nu\sigma\kappa} (M_{\sigma} F_{\kappa\lambda} + M_{\kappa} F_{\lambda\sigma}) = 0 \quad 4.1.11$$

or

$$F_{\kappa\lambda} \chi^{\lambda\nu\sigma\kappa} M_{\sigma} + F_{\lambda\sigma} \chi^{\lambda\nu\sigma\kappa} M_{\kappa} = 0 \quad 4.1.12$$

Now we apply a tensor property to have

$$F_{\lambda\kappa} \chi^{\lambda\nu\sigma\kappa} M_{\sigma} + F_{\lambda\sigma} \chi^{\lambda\nu\sigma\kappa} M_{\kappa} = 0 \quad 4.1.13$$

Then we can interchange two of the dummy variables to realize

$$F_{\lambda\sigma} \chi^{\lambda\nu\sigma\kappa} M_{\kappa} + F_{\lambda\sigma} \chi^{\lambda\nu\sigma\kappa} M_{\kappa} = 0 \quad 4.1.14$$

or

$$F_{\lambda\sigma} \chi^{\lambda\nu\sigma\kappa} M_{\kappa} = 0 \quad 4.1.15$$

In order to have a form which will be in accord with the classical expression, we multiply by M_{λ} (This, of course, does not affect the equivalence of our expression.) i.e.,

$$F_{\lambda\sigma} M_{\lambda} \chi^{\lambda\nu\sigma\kappa} M_{\kappa} = 0 \quad 4.1.16$$

Since the field quantities are not zero on the boundary the determinant of the coefficients of the field quantities must be zero, i.e.,

$$| M_{\lambda} \chi^{\lambda\nu\sigma\kappa} M_{\kappa} | = 0 \quad 4.1.17$$

where $\lambda, \nu, \sigma, \kappa = 0, 1, 2, 3$. This is the tensor form of the generalized eiconal equation. It is generally reducable to an equation of the fourth degree which results in four solutions: two wave fronts traveling away from the source and two wave fronts toward the source.

4.2 Eiconal Equation for the Diffraction Medium

Let us limit the development to three variables for illustration.

Assume then that the solution is independent of y , so that

$$\psi = \psi(x, z, t) \quad 4.2.1a$$

and

$$M_{\lambda} = (M_0, M_1, 0, M_3) \quad 4.2.1b$$

We then introduce the general eiconal equation

$$| M_{\lambda} \chi^{\lambda \nu \sigma \kappa} M_{\kappa} | = 0 \quad 4.1.17$$

and expand for $\nu, \sigma = 0, 1, 2, 3$ which becomes

$$\begin{vmatrix} B^{00} & B^{01} & B^{02} & B^{03} \\ B^{10} & B^{11} & B^{12} & B^{13} \\ B^{20} & B^{21} & B^{22} & B^{23} \\ B^{30} & B^{31} & B^{32} & B^{33} \end{vmatrix} = 0 \quad 4.2.2$$

We now expand on $\lambda, \kappa = 0, 1, 2, 3$ for each element, e.g.,

$$\begin{aligned} B^{00} &= M_{\lambda} (\chi^{\lambda 000} M_0 + \chi^{\lambda 001} M_1 + \chi^{\lambda 002} M_2 + \chi^{\lambda 003} M_3) \\ &= M_0 (\chi^{0000} M_0 + \dots) \dots + M_3 (\chi^{3000} M_0 + \dots) \end{aligned} \quad 4.2.3$$

We have by inspection of the constitutive tensor equation (2.2.10)

$$B^{00} = M_1 \chi^{1001} M_1 + M_3 \chi^{3003} M_3 \quad 4.2.4$$

We continue in a similar manner for the other elements and we arrive at the determinant shown in Figure 3. Since $M_0 \neq 0$, let

$$u_1 = \frac{M_1}{M_0} \quad 4.2.5$$

$$u_3 = \frac{M_3}{M_0} \quad 4.2.6$$

and apply the property $\chi^{1001} = -\chi^{0101}$, et., to have the simplified determinant shown in Figure 4.

$$\begin{vmatrix} M_1 X^{1001} M_1 + M_3 X^{3003} M_3 & M_1 X^{1010} M_0 & 0 & M_3 X^{3030} M_0 \\ M_0 X^{0101} M_1 & M_0 X^{0110} M_0 + M_3 X^{3113} M_3 & 0 & M_3 X^{3131} M_1 \\ 0 & 0 & M_0 X^{0220} M_0 + M_1 X^{1221} M_1 + M_3 X^{3223} M_3 & 0 \\ M_0 X^{0303} M_3 & M_1 X^{1313} M_3 & 0 & M_0 X^{0330} M_0 + M_1 X^{1331} M_1 \end{vmatrix} = 0$$

Figure 3. Determinant of the Eiconal

$-u_1^2 x^{0202} - u_3^2 x^{0303}$	$u_1 x^{0101}$	0	$u_3 x^{0303}$
$u_1 x^{0101}$	$-x^{0101} - u_3^2 x^{3131}$	0	$u_1 u_3 x^{3131}$
0	0	$-x^{0202} - u_1^2 x^{1212} - u_3^2 x^{2323}$	0
$u_3 x^{0303}$	$u_1 u_3 x^{3131}$	0	$-x^{0303} - u_1^2 x^{3131}$

= 0.

Figure 4. Simplified Determinant of the Eiconal

The determinant can be shown to be identically zero; however, there exists subdeterminants of order three not identically zero, which is to say that the eiconal equation is over-constrained.

The electromagnetic field is gage invariant, i.e., the four-potentials are nothing more than useful mathematical auxiliaries which can be modified at will, leaving the field vectors \vec{E} and \vec{B} undisturbed. We can choose a gage such that the fields on the boundary are determined by three of the potentials instead of four.

Inspection of a cofactor expansion of the 4×4 determinant will reveal that the resulting three non-zero 3×3 subdeterminants reduce to identical expressions. Therefore, we can set the subdeterminant expression equal to zero, and solve for the unique u 's which will satisfy a nontrivial solution, i.e.,

$$\begin{vmatrix}
 u_1 \chi^{0101} & 0 & u_3 \chi^{0303} \\
 -\chi^{0202} - u_3^2 \chi^{3131} & 0 & u_1 u_3 \chi^{3131} \\
 0 & -\chi^{0202} - u_1^2 \chi^{1212} - u_3^2 \chi^{2323} & 0
 \end{vmatrix} = 0$$

4.2.7

which reduces to

$$u_1^2 \chi^{0101} \chi^{3131} + \chi^{0101} \chi^{0303} + u_3^2 \chi^{0303} \chi^{3131} = 0 .$$

4.2.8

If we substitute for the constitutive tensor elements from equation (2.2.10), we have

$$u_1^2 + u_1^2 = \mu\epsilon \quad 4.2.9$$

or

$$\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial z}\right)^2 = \frac{\mu\epsilon}{c^2} \left(\frac{\partial\Psi}{\partial t}\right)^2 \quad 4.2.10$$

which for the diffraction medium is really

$$\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial z}\right)^2 = \frac{\mu\epsilon}{c^2} [1 + m \cos(\omega t - \beta z)] \left(\frac{\partial\Psi}{\partial t}\right)^2 \quad 4.2.11$$

In a similar fashion, the general eiconal equation for the diffraction medium can be shown to be

$$\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2 + \left(\frac{\partial\Psi}{\partial z}\right)^2 = \frac{\mu\epsilon}{c^2} [1 + m \cos(\omega t - \beta z)] \left(\frac{\partial\Psi}{\partial t}\right)^2 \quad 4.2.12$$

We now proceed to find a nonlinear transformation whereby the constitutive tensor remains invariant (independent of the coordinates) or we might say that our inhomogeneous scalar $\epsilon[1+m \cos(\omega t - \beta z)]$ becomes the homogeneous scalar ϵ .

We assume the following

$$\begin{aligned} x^{0'} &= x^0 \\ x^{1'} &= x^1 \\ x^{2'} &= x^2 \\ x^{3'} &= \omega x^0 - \beta x^3 \end{aligned} \quad 4.2.13$$

Evaluating the Jacobian, equation (2.3.1), we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & -\beta \end{bmatrix} \quad 4.2.14$$

Then

$$\Delta^{-1} = -1/\beta \quad 4.2.15$$

and

$$\chi^{\lambda' \nu' \sigma' \kappa'} = \Delta^{-1} A_{\lambda \nu \sigma \kappa}^{\lambda' \nu' \sigma' \kappa'} \chi^{\lambda \nu \sigma \kappa} \quad 2.2.8$$

On evaluating from equation (4.2.14) and (2.2.10), we have

$$\chi^{\lambda' \nu' \sigma' \kappa'} = \begin{bmatrix} \frac{\epsilon'}{\beta} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon'}{\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta \epsilon' & 0 & 0 & 0 \\ 0 & 0 & 0 & (\frac{\omega^2 \epsilon'}{\beta} + \frac{\beta}{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 & (\frac{\omega^2 \epsilon'}{\beta} + \frac{\beta}{\mu}) & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\beta \mu} \end{bmatrix} \quad 4.2.16$$

where

$$\epsilon' = \epsilon [1 + m \cos x^{3'}] \quad 4.2.17$$

Now we substitute into the eiconal, equation (4.2.8), and attempt to solve for Ψ . The method is a trial and error technique and requires considerable intuition of the problem. Let us now attack the problem by more conventional means to gain insight of the wave fronts.

CHAPTER V

WAVE FRONTS FROM RAYS

5.1 Rays

The energy of the geometrical optics field moves along the orthogonal trajectories to the family of wave fronts which in isotropic media are called the rays or bicharacteristics of the wave fronts. In isotropic media $\bar{D} = \epsilon \bar{E}$ and $\bar{B} = \mu \bar{H}$, where ϵ and μ are scalars. Generally in isotropic media they are functions of x , y , z , and t making the media inhomogeneous which is true for our case. The surface $\psi(t, x, y, z) = 0$ from which the wave fronts are derived are the surfaces on which discontinuities of the second or first derivative of the solution function $u(t, x, y, z)$ of Maxwell's equations exist, and the rays are the curves along which the discontinuities propagate.

We know that M_0 or ψ_t is not zero, so we can solve $\psi(t, x, y, z) = 0$ explicitly for t and write the equation of the family of wave fronts as

$$\phi(x, y, z) = ct \quad 5.1.1$$

where we have chosen the constant c , the velocity of light in a vacuum, in our definition of ϕ . This choice will become clear in the development. This means that the surface $\psi = 0$ can be expressed in the form

$$\psi(t, x, y, z) = \phi(x, y, z) - ct = 0 \quad 5.1.2$$

For this form of ψ the general eiconal equation (4.2.12) becomes

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 = \epsilon \mu \quad 5.1.3$$

where $\epsilon \mu$ is representative of the medium behavior for a general case, that is, $\epsilon \mu$ may be a function of (x, y, z) as in the diffraction problem.

Traditionally bicharacteristics refer to curves in the (t, x, y, z) space and lying on $\psi(t, x, y, z) = 0$, that is, the solutions of the ordinary differential equations corresponding to the characteristic condition regarded as a partial differential equation in the four variables. Rays refer to curves in (x, y, z) space and orthogonal to $\phi(x, y, z) = ct$, that is, solutions of the ordinary differential equations corresponding to the classical eiconal differential equation. Geometrically, the rays in this sense are the projections of the bicharacteristics onto the (x, y, z) space.

The radiation vector on a wave front or the geometrical optics energy flux density is defined

$$\bar{S}^* = \frac{c}{4\pi} (\bar{E}^* \times \bar{H}^*) \quad 5.1.4$$

where \bar{S}^* is the value observed at (x, y, z) at time $t = \frac{1}{c} \phi(x, y, z)$.

Since in isotropic media \bar{E}^* and \bar{H}^* are both perpendicular to $\text{grad } \phi$, S^* has the direction of $\text{grad } \phi$. It can be shown (Kline) using the vector triple product identity that

$$\bar{S}^* \cdot \text{grad } \phi = c \bar{W}^* \quad 5.1.5$$

Where W^* denotes the total geometrical optics energy density on a wave front. Then since a unit vector in the direction of $\text{grad } \phi$ is defined

by equation (4.1.7a), we have

$$\Gamma = | \text{grad } \phi | = \sqrt{\epsilon\mu} = n \quad 5.1.6$$

It follows that

$$\bar{S}^* \cdot \frac{\text{grad}}{\Gamma} = \frac{c\bar{W}^*}{n} \quad 5.1.7$$

and because \bar{S}^* has the direction of $\text{grad } \phi$

$$\bar{S}^* = \frac{c\bar{W}^*}{n^2} \text{grad } \phi \quad 5.1.8$$

It then follows

$$| \bar{S}^* | = \frac{c\bar{W}^*}{n} \quad 5.1.9$$

The ratio of c to the index of refraction, n , of the medium is the velocity with which the wave fronts propagate through the medium, that is,

$$v = \frac{c}{n} \quad 5.1.10$$

where v is the velocity of the wave fronts. By equation (5.1.9) the magnitude of the flux density is the energy density on the wave front multiplied by the velocity with which the front moves through the medium.

It is natural then to introduce the family of orthogonal trajectories to the family of wave fronts $\phi = ct$ as the curves along which \bar{E}^* , \bar{H}^* and the energy \bar{S}^* flow. These curves are the electromagnetic rays.

5.2 Method of Solution

In the physical x -space the characteristic condition $H(\partial_\lambda \psi g^{\lambda\sigma}, x^\lambda) = 0$ for the normals $\partial_\lambda \psi g^{\lambda\sigma} = (\text{grad } \psi)_\lambda$ must be interpreted in terms of ray surfaces $\psi(x^\lambda) = \text{const.}$, $\lambda = 0, 1, \dots, n$. Since $H = 0$ is a partial differential equation of first order for $\psi(x^\lambda)$, then all members of the family $\psi = \text{const.}$ are ray surfaces.

Every solution of the characteristic partial differential equation $H = 0$ is generated by an n -parameter family of characteristic curves or rays belonging to the first-order equation $H = 0$. These rays are supplemented to ray strips by attaching to them the values of

$$M_\lambda = \frac{\partial_\lambda \psi}{\Gamma} g^{\lambda\sigma} \quad 4.1.7a$$

(We consider the values M_λ along such a curve as functions of a curve parameter τ , and we shall denote by a dot differentiation with respect to τ). Then the ray strips satisfy the system of $2n + 2$ canonical ordinary differential equations

$$\dot{x}^\lambda = \gamma H M_\lambda \quad 5.2.1a$$

$$\dot{M}_\lambda = -\gamma H_{x^\lambda} \quad 5.2.1b$$

for which $H(x^\lambda, M_\lambda)$ is an integral, and γ is an arbitrary function of x^λ . We impose the condition $H = 0$ at one point of each ray; then the above two equations together with $H(x^\lambda, M_\lambda) = 0$ define a $2n$ -parameter family of ray strips which is determined independently of the specific ray surfaces $\psi = \text{const.}$

Now consider the simplified eiconal equation (5.1.3). Let us assume for a moment that we have a $\phi(x^\lambda)$ which is a solution of $H(x^\lambda, M_\lambda) = 0$. Then the surfaces $\phi = \text{const.}$ determine a one-parameter family of wave fronts and a two-parameter family of rays. We let $x^\lambda = x^\lambda(\tau)$ be one of these rays and let P_0 , which corresponds to $\tau = 0$, and P , which corresponds to a general value of τ , be two points on this ray. Then

$$\phi(P) - \phi(P_0) = \int_0^\tau \frac{d\phi}{d\tau} d\tau \quad 5.2.2$$

Since along this ray $\phi = \phi(x^\lambda(\tau))$, then

$$\phi(P) - \phi(P_0) = \int_0^\tau \left(\sum_\lambda M_\lambda \dot{x}^\lambda \right) d\tau \quad 5.2.3$$

5.3 Solution

Since our purpose is to present a method of solution rather than solve the problem explicitly, let us make an approximation. Since the difference in the velocities of sound and light is on the order of 10^5 , we realize that the term ωt in equation (4.2.11) is negligible. Then from equations (5.1.2) and (5.1.3) and using the procedure in section (5.2) the eiconal equation for the diffraction medium may be written

$$H = (M_1)^2 + (M_3)^2 - \mu\epsilon(1 + m \cos \beta x^3) = 0 \quad 5.3.1a$$

or

$$(M_1)^2 + (M_3)^2 = n^2(x^3) \quad 5.3.1b$$

where

$$n^2(x^3) = \mu\epsilon(1 + m \cos \beta x^3) \quad 5.3.1c$$

Then using equations (5.2.1a) and 5.2.1b) with $\gamma = 1/2$, we have a system of differential equations for the rays

$$\begin{aligned} \dot{M}_1 &= 0 & \dot{x}^1 &= M_1 \\ \dot{M}_3 &= 1/2(n^2)_{x^3} & \dot{x}^3 &= M_3 \end{aligned} \quad 5.3.2$$

where the dot denotes differentiation with respect to the parameter τ . Each ray will then be given in parametric form by a solution of the above, namely,

$$x^1(\tau), x^3(\tau), M_1(\tau), M_3(\tau)$$

subject to the initial conditions that a ray must pass through a selected point $[(x^1)_0, (x^3)_0]$ with a selected initial direction given by the direction numbers $[(M_1)_0, (M_3)_0]$, that is,

$$\begin{aligned} (x^1)_0 &= x^1(0) & (M_1)_0 &= M_1(0) \\ (x^3)_0 &= x^3(0) & (M_3)_0 &= M_3(0) \end{aligned} \quad 5.3.3$$

From a vacuum the incident electric field will have the form

$$\overline{E} = \overline{E}_0 e^{ik_0\psi} = \overline{E}_0 e^{ik_0(\phi - ct)} \quad 5.3.4$$

where k_0 is the wave number

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\nu}{c} \quad 5.3.5$$

The length λ_0 is the wave length and ν is the angular frequency for light. Our initial wave front will originate at the boundary so that

$$\phi_{init} = n_0 x^1 \quad 5.3.6$$

where n_0 is the index of refraction for a vacuum or $n_0 = \sqrt{\epsilon\mu}$.

For convenience, let $\tau = 0$ be a point on the boundary, whence

$$\phi_0 = n_0 (x^1)_0 \quad 5.3.7$$

The direction numbers at this point from equation (5.3.6) are

$$(M_1)_0 = n_0 \quad 5.3.8$$

$$(M_3)_0 = 0$$

Equations (5.3.2) can be solved up to quadratures by means of the observation that M_1 is a constant. It follows from equation (5.3.8) that

$$x^1(\tau) = n_0 \tau + (x^1)_0 \quad 5.3.9$$

The second ray equation in (5.3.2) can be written

$$\frac{d}{d\tau} [(\dot{x}^3)^2] = \frac{d(n^2)}{dx^3} \dot{x}^3 = \frac{d(n^2)}{d\tau} \quad 5.3.10$$

We integrate equation (5.3.10) from 0 to τ , obtaining

$$(\dot{x}^3)^2(\tau) - (\dot{x}^3)^2(0) = n^2(\tau) - n_0^2 \quad 5.3.11$$

which, after substitutions are made from equations (5.3.2), (5.3.3), and (5.3.8) is equivalent to

$$\dot{x}^3(\tau) = \pm \sqrt{n^2(x^3) - n_0^2} \quad 5.3.12$$

We now divide through by $\pm \sqrt{n^2(x^3) - n_0^2}$ and integrate from $\tau = 0$ to τ

(or from $x^3 = (x^3)_0$ to x^3) and arrive at

$$\pm \tau = \int_{(x^3)_0}^{x^3} \frac{dx^3}{\sqrt{n^2(x^3) - n_0^2}} \quad 5.3.13$$

In equation (5.3.13) we select the plus sign so that x^3 can increase with increasing τ , for it is convenient to have the wave propagate in the direction of increasing x^3 . Our limitation to an infinite half-space ensures that equation (5.3.13) will be unambiguous, i.e., no reflected waves.

From equations (5.3.9) and (5.3.13) we obtain an equation for any ray

$$x^1 = (x^1)_0 + \int_{(x^3)_0}^{x^3} \frac{n_0 dx^3}{\sqrt{n^2(x^3) - n_0^2}} \quad 5.3.14$$

or substituting from equation (5.3.1c)

$$x^1 = (x^1)_0 + \int_{(x^3)_0}^{x^3} \frac{dx^3}{\sqrt{m \cos \beta x^3}} \quad 5.3.15$$

The solution ϕ of the eiconal equation (5.1.3) can now be determined explicitly. According to equation (5.2.3) and in view of equations (5.3.1b) and (5.3.2) we have

$$\phi = \phi_0 + \int_0^\tau n^2[x^3(\tau)] d\tau \quad 5.3.16$$

From equations (5.3.7), (5.3.13), and (5.3.16) we obtain

$$\phi = n_0(x^1)_0 + \int_{(x^3)_0}^{x^3} \frac{n^2(x^3) dx^3}{\sqrt{n^2(x^3) - n_0^2}} \quad 5.3.17$$

Using equation (5.3.14), we then find

$$\phi = n_0 x^1 + \int_{(x^3)_0}^{x^3} \frac{\sqrt{n^2(x^3) - n_0^2}}{n_0} dx^3 \quad 5.3.18$$

or substituting from equation (5.3.1c)

$$\phi = n_0 x^1 + n_0 \int_{(x^3)_0}^{x^3} \frac{\sqrt{m \cos \beta x^3}}{m} dx^3 \quad 5.3.19$$

The two terms of equation (5.3.19) can be interpreted as the eiconal corresponding to our initial plane wave plus a correction due to the inhomogeneous nature of the medium.

From equation (5.1.2), ψ may be expressed

$$\psi = n_0 x^1 + \int_{(x^3)_0}^{x^3} \sqrt{n^2(x^3) - n_0^2} dx^3 - cx^0 \quad 5.3.20$$

which is the solution to the eiconal equation (4.2.11) of the diffraction medium. This can be verified by direct substitution if we remember that $n^2(x^3) = \mu\epsilon(1 + m \cos \beta x^3)$, where ωx^0 has been neglected.

The rays are plotted in Figure 5 using equation (5.3.15). We observe that the rays are spaced 2π radians apart for any given time; this is equivalent to 300 microns for a one megahertz sound wave. Distortion of the wave front increases as the wave progresses deeper into the medium. The wave front, ϕ , is plotted in Figure 6 using equation (5.3.19). Inspection of this equation as well as equation (5.3.15) indicates that the rays and the wave front become complex for $\frac{\pi}{2} < \beta x^3 < \frac{3\pi}{2}$. The modulus of this complex refractive index is interpreted (Born) as absorption or damping as indicated by the dotted curves in Figure 6. Plotting was facilitated using numerical integration by Simpson's 1/3 rule on a digital computer. Constants used were as follows:

Sound velocity, dry air = 331 m/sec

Frequency of Ultrasound = 1 megahertz

Modulation Index, $m = 0.25$

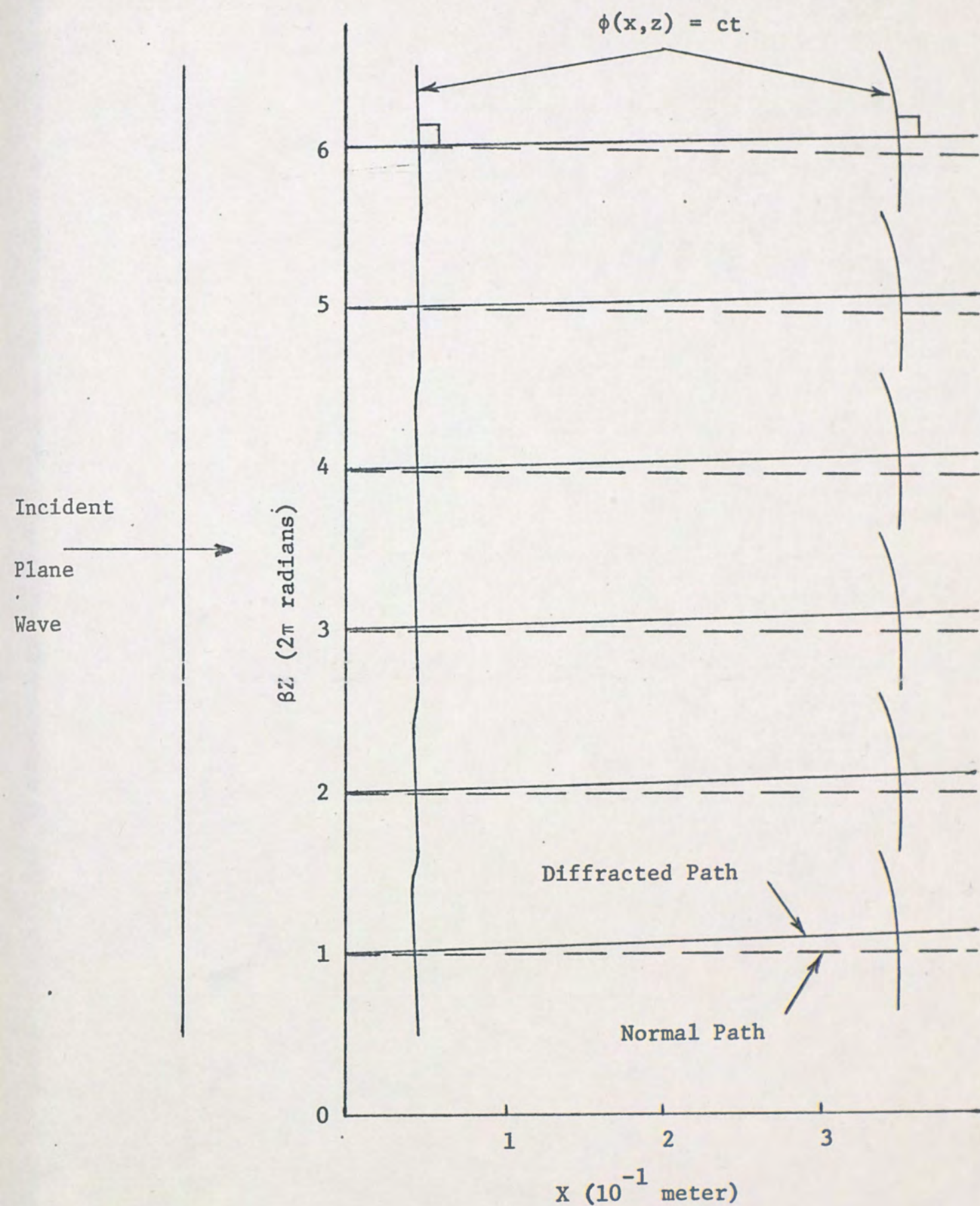


Figure 5. Rays in the Diffraction Medium

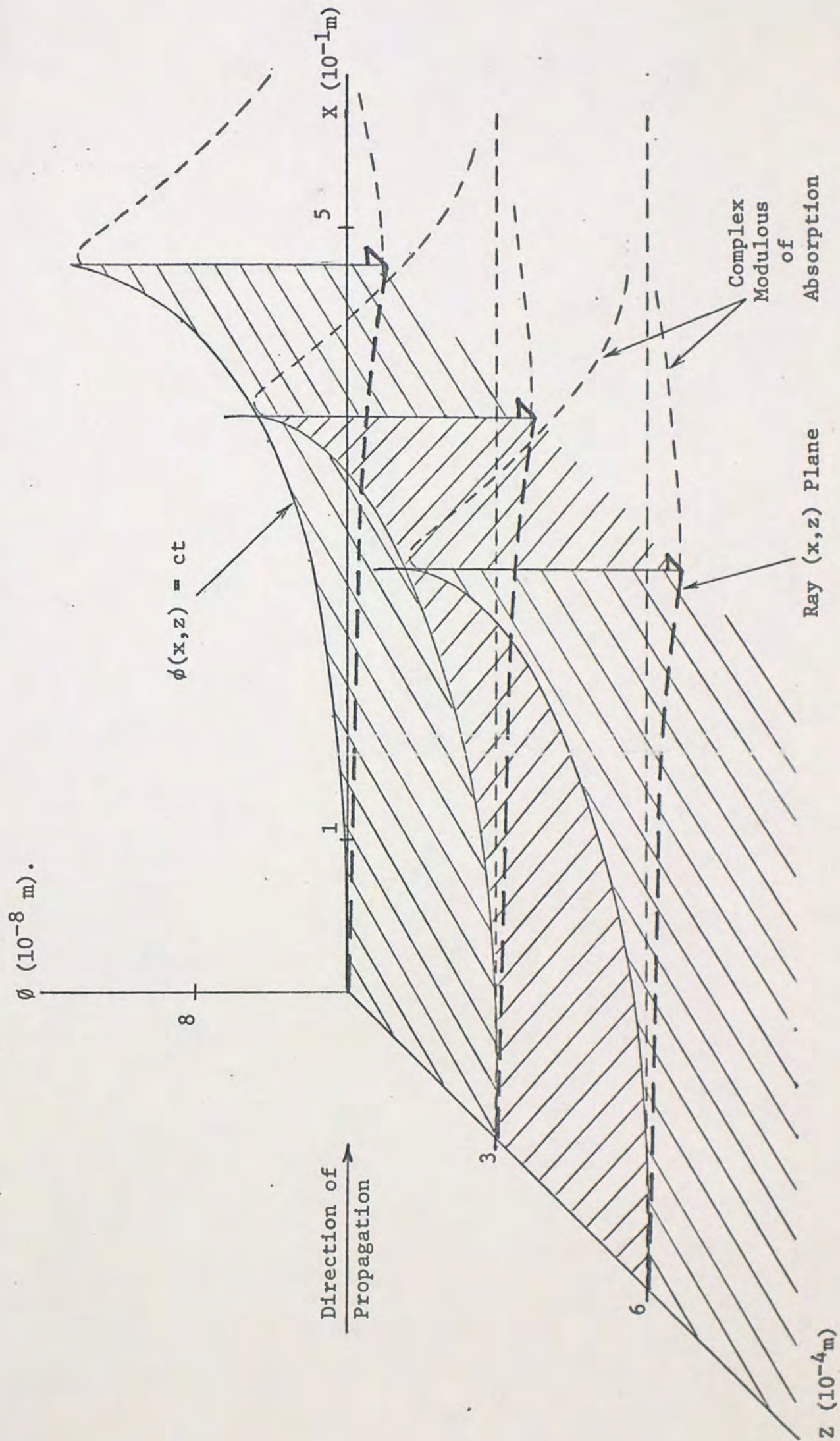


Figure 6. Wave Front in the Diffraction Medium

CHAPTER VI

CONCLUSION

6.1 Summary

We have presented a development of a covariant form of Hamilton's Principle of Stationary Action and observed its usefulness in investigating the formal structure of electromagnetics, in specific, diffraction of light by ultrasound. A unique space-time manifold where time was treated as an equal parameter with space was introduced. The electric scalar potential and the magnetic scalar potential were combined into a single four-component potential allowing us to formulate a covariant tensor action integral. We investigated the variation of the action integral and found the equations that the four-potential must satisfy on the boundary of the domain. The boundary equations were utilized to develop the general eiconal equation which was then expanded for the diffracted medium and shown equivalent to the conventional form.

We then investigated the rays which are the geometrical lines along which the wave fronts travel in the medium. A solution of the eiconal for the diffracted medium was successfully made by introducing a new parameter and performing a canonical transformation.

6.2 Conclusions

In general, we can conclude that Hamilton's Principle of Stationary Action is a useful tool in investigating the formal

structure of electromagnetics. We observed the insight that it gave to the relationship between physical and geometrical optics. We also experienced the convenience of the tensor form of the equations in formulating problems in any desired coordinate system.

More specifically, we can conclude that the investigation of Hamilton's Principle has shown the insight that geometrical optics gives in electromagnetic problems. The fundamental eiconal equation of geometrical optics is directly related to Maxwell's equations of electromagnetics. Thus we have a tool whereby we are able to study and solve the relatively invisible problems of electromagnetics. From the eiconal equation we can derive the rays or geometrical lines along which the wave fronts and consequently the energy of the propagating source is moving. We are thus able to construct the effects of various media on the propagating wave via the eiconal and the associated rays.

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